2016 Summer Project Report

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I worked with Julio in 2016 summer to find upper bounds for the change in price after modifying parameters of the Black Scholes equation. The following is a brief report of our progress.

1. Part 1

   In solving the Black-Scholes Equation for European Options, we need to solve:

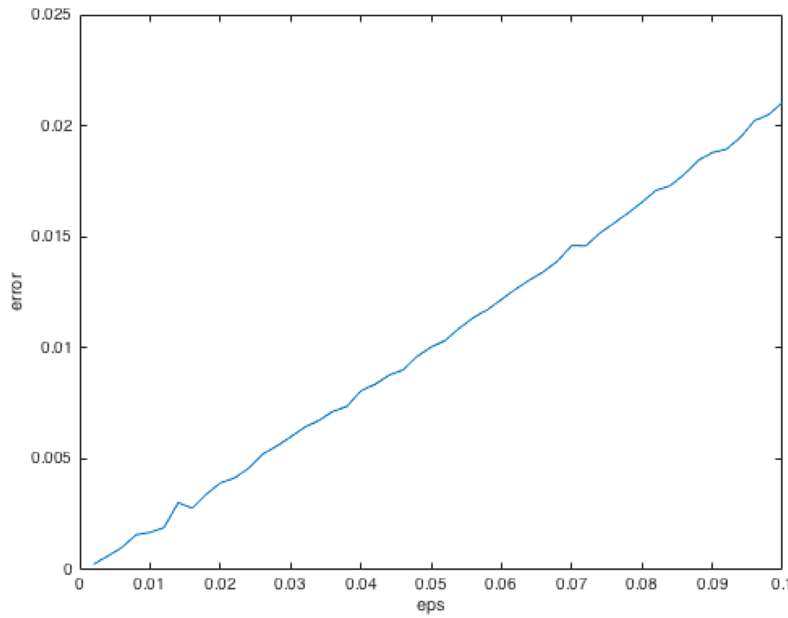
   $$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j} \rho_{ij}\sigma_i\sigma_j s_i s_j \frac{\partial^2 V}{\partial s_i \partial s_j} + r\sum_i s_i \frac{\partial V}{\partial s_i} - rV = 0,$$

   with the solution:

   $$V(s_t, t) = \frac{e^{(-r(T-t))}}{(2\pi)^{\frac{n}{2}}|detA|^{\frac{1}{2}}} \int_{R^n} \Phi(s_t \cdot e^x) exp(-\frac{1}{2}\alpha^T \cdot A^{-1} \cdot \alpha) dx \text{ [1]},$$

   where $\alpha_i = (x_i - (r - \frac{\sigma_i^2}{2})(T-t)), A = (T-t)(M \cdot C \cdot M)$,C is the correlation matrix and M is the matrix with σ_i on its diagonal and zero elsewhere. $\Phi(s) = max(\sum_{i=1}^n c_i s_i - K), 0)$ is the payoff function of the Basket option.

   Now we try to obtain the error of $|V(s, A) - V(s, B)|$ where A is the correlation matrix of a 2-D Option and B is constructed by adding small ϵ to the eigenvalues of A. The following plot is the result based on Monte Carlo Simulation: $(s_1(0) = s_2(0) = 1, K = 0.92, \sigma_1 = \sigma_2 = 0.2, corr = 0.3, r = 0.02, T = 1)$



The plot show a promising linear relationship between eps and abs(error). Therefore, it is reasonable to try bounding the error in O(eps). The value of error in the plot is listed as below:

	A	B	C	D	E	F	G	H	I	J	K
1	eps	0.0001	0.0002	0.0003	0.0004	0.0005	0.0006	0.0007	0.0008	0.0009	0.001
2	abs(error)	0.00165174	0.00151973	0.00107408	0.00223727	0.00048755	0.00083981	0.00094403	0.00110867	0.00142016	0.00083939
3	eps	0.0011	0.0012	0.0013	0.0014	0.0015	0.0016	0.0017	0.0018	0.0019	0.002
4	abs(error)	0.00046817	0.00067393	0.00055831	0.00154926	0.00173908	0.00088704	0.00282165	0.00046213	0.00292849	0.0038816
5	eps	0.0021	0.0022	0.0023	0.0024	0.0025	0.0026	0.0027	0.0028	0.0029	0.003
6	abs(error)	0.00172313	0.00264315	0.00251311	0.00378035	0.00059991	0.00261149	0.00113479	0.00051657	0.00174619	0.00099863
7	eps	0.0031	0.0032	0.0033	0.0034	0.0035	0.0036	0.0037	0.0038	0.0039	0.004
8	abs(error)	0.00387764	0.00508325	0.00501587	0.00222668	0.00289314	0.00321627	0.00350926	0.00294951	0.00438057	0.00171726
9	eps	0.0041	0.0042	0.0043	0.0044	0.0045	0.0046	0.0047	0.0048	0.0049	0.005
10	abs(error)	0.00125787	0.00240314	0.00438747	0.00174451	0.00391971	0.00333395	0.00221228	0.00416647	0.00523315	0.00311729
11	eps	0.0051	0.0052	0.0053	0.0054	0.0055	0.0056	0.0057	0.0058	0.0059	0.006
12	abs(error)	0.00275968	0.00360068	0.00429316	0.00560536	0.00664018	0.00583883	0.00507786	0.00192142	0.00518354	0.00547113
13	eps	0.0061	0.0062	0.0063	0.0064	0.0065	0.0066	0.0067	0.0068	0.0069	0.007
14	abs(error)	0.00336384	0.00598693	0.00551183	0.00405403	0.00754181	0.00619137	0.00819842	0.00236189	0.00627422	0.00792103
15	eps	0.0071	0.0072	0.0073	0.0074	0.0075	0.0076	0.0077	0.0078	0.0079	0.008
16	abs(error)	0.00536911	0.00615592	0.00610892	0.00443998	0.00507223	0.00590069	0.00663009	0.00488026	0.00582131	0.00691167
17	eps	0.0081	0.0082	0.0083	0.0084	0.0085	0.0086	0.0087	0.0088	0.0089	0.009
18	abs(error)	0.00870377	0.00805364	0.01090508	0.00837291	0.00816469	0.0080943	0.00521022	0.00820542	0.00739848	0.00792799
19	eps	0.0091	0.0092	0.0093	0.0094	0.0095	0.0096	0.0097	0.0098	0.0099	0.01
20	abs(error)	0.00618524	0.00496294	0.00701087	0.00868313	0.01095315	0.00840718	0.0095743	0.00749043	0.00799812	0.00992608

2. Part 2
   In this part, we intend to focus on one dimensional case to better understand the effect of changing volatility on the difference in the

prices. In one dimension, [1] can be simplified into the following:

$$V(s_t,t) = \frac{e^{(-r(T-t))}}{(2\pi)^{\frac{1}{2}}(T-t)^{\frac{1}{2}}\sigma}\int_R \Phi(s_t\cdot e^x)exp(-\frac{1}{2}\cdot\frac{a^2}{(T-t)\sigma^2})dx \ [2],$$

where $a = x-(r-\frac{\sigma^2}{2})(T-t)$. If we let $C(t) = e^{(-r(T-t))}$ and $P_\sigma(x) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}(T-t)^{\frac{1}{2}}}exp(-\frac{1}{2}\cdot\frac{a^2}{(T-t)\sigma^2})$, then

$$V(s_t,t) = C(t)\int_R \Phi(s_t\cdot e^x)P_\sigma(x)dx \ [3]$$

For Put Options, payoff function $\Phi(s_t\cdot e^x) = max(K-s_t\cdot e^x, 0)$. We can obtain an explicit form of $V(s_t,t)$.

Consider $I(\sigma) = \int_R \Phi(s_t\cdot e^x)P_\sigma(x)dx$, this is the part where different σ contributes to difference in prices. Therefore, we want to change the value of σ from 0 to large to see the role σ plays on the price. Note that when $\sigma\to 0$, $P_\sigma(x) = \delta_m(x)$. Define

$$E(\sigma) = \int_R \Phi(s_t\cdot e^x)(P_\sigma(x)-\delta_m(x))dx = \int_R \Phi(s_t\cdot e^x)P_\sigma(x)dx - F(m).$$

Let

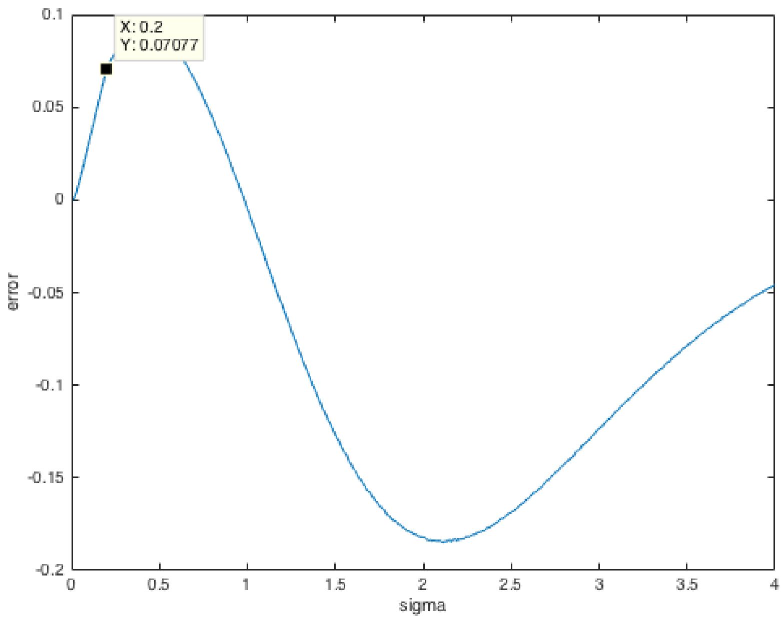
$$\frac{d_1 = log(S/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$\frac{d_2 = log(S/K)+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t},}$$

$$E(\sigma) = \int_R \Phi(s_t\cdot e^x)P_\sigma(x)dx - F(m)$$

$$= K\Phi(-d_2) - s_t e^{r(T-t)}\Phi(-d_1) - (K-s_t e^m)_+$$

The next plot is based on setting$(K = 1, r = 0.1, s = 1, T-t = 1)$



Note that before $\sigma = 0.2$, the error is linear with respect to σ. This is because our payoff function becomes zero when $\sigma < 0.2$.

3. Part 3

In this part, we try to obtain a upper bound for the difference. Recall [1] in Part 1, if we eigen-decompose $A = Q^T D_a Q$, $V(s_t, t)$ can be rewrite into a function of $a = [a_1, a_2, ..., a_n]$, where $a_i = eigenvalue(A)_i$.

$$G(a) = \frac{e^{(-r(T-t))}}{(2\pi)^{\frac{n}{2}} |detA|^{\frac{1}{2}}} \int_{R^n} f(x) exp(-\frac{1}{2}(\alpha^T \cdot (Q^T D_a Q)^{-1} \cdot \alpha) dx \ [4]$$

If we substitute $y = Q^{-1}\alpha$ and $dx = Qdy$. Again, $|detA|^{\frac{1}{2}} = |det(Q^T D_a Q)|^{\frac{1}{2}} = (\prod_{i=1}^n a_i)^{\frac{1}{2}}$. [4] can be rewrite into

$$G(a) = \frac{e^{(-r(T-t))}}{(2\pi)^{\frac{n}{2}} |\prod_{i=1}^n a_i|^{\frac{1}{2}}} \int_{R^n} f(Qy + m) exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y)) Qdy \ [5].$$

Now the difference of $|G(a) - G(B)|$ can be bounded by applying Mean Value Theorem:

$$|G(a) - G(b)| = \nabla G(\theta) \cdot (b - a), \ \theta \in (a, b)$$

$$= \sum_i \partial_i G(\theta)(b_i - a_i) \leq \max_i \partial_i G(\theta) * |b - a|.$$

Therefore, we only need to bound $\partial_i G(\theta)$. By [5], we can calculate the derivative of G:

$$\partial_j G(a) = \int_{R^n} \partial_i [\frac{e^{(-r(T-t))}}{(2\pi)^{\frac{n}{2}} (\prod_{i=1}^n a_i)^{\frac{1}{2}}} f(Qy + m) exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y))Q]dy$$

$$= \int_{R^n} \frac{e^{(-r(T-t))} f(Qy + m)}{(2\pi)^{\frac{n}{2}}} \partial_j [(\prod_{i=1}^n a_i)^{-\frac{1}{2}} exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y))]Qdy. [6]$$

Let $K = \partial_j [|\prod_{i=1}^n a_i|^{-\frac{1}{2}} exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y))]$, then

$$K = \partial_j [(\prod_{i=1}^n a_i)^{-\frac{1}{2}}] exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y)) + (\prod_{i=1}^n a_i)^{-\frac{1}{2}} \partial_j [exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y))]$$

$$= -\frac{1}{2a_j} * (\prod_{i=1}^n a_i)^{-\frac{1}{2}} * exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y))$$

$$+ (\prod_{i=1}^n a_i)^{-\frac{1}{2}} * exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y)) * \partial_j [-\frac{1}{2} y^t D_a^{-1} \cdot y].$$

Denote $J = \partial_j [y^t D_a^{-1} \cdot y]$ and as $D_a^{-1} = Diag(a_i^{-1})$,

$$J = \partial_j [-\frac{1}{2} \sum_{i=1}^n a_i^{-1} y_i^2] = \frac{1}{2} * \frac{y_j^2}{a_j^2}.$$

Therefore,

$$K = -\frac{1}{2a_j} * (\prod_{i=1}^n a_i)^{-\frac{1}{2}} * exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y)) + (\prod_{i=1}^n a_i)^{-\frac{1}{2}} * exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y)) * \frac{1}{2} * \frac{y_j^2}{a_j^2}.$$

Back to [6],

$$\partial_j G(a) = \int_{R^n} \frac{e^{(-r(T-t))} f(Qy + m)}{(2\pi)^{\frac{n}{2}}} \{-\frac{1}{2} * (\prod_{i=1}^n a_i)^{-\frac{1}{2}} * exp(-\frac{1}{2}(y^t D_a^{-1} \cdot y)) * (\frac{1}{a_j} - \frac{y_j^2}{a_j^2})\} Qdy. [7]$$

4. Conclusion

In this project, I helped in numerical estimation of the behaviour of two circumstance in Part 1 and 2. In part 3, I helped derived the derivative of function G, which can be used in finding upper bound for the difference when change correlation of the stocks. Due to the limited time I have working with Julio, I did not complete the research on the [7] of Part 3. In further discussion, I would exam more carefully on the nature on [7] to bound the derivative. More numerical estimation can also be applied to the integral for a better sense of the function.

<u>APPENDIX:</u>

Part 1(pricing.m)

```
function [ V ] = pricing( A )
%UNTITLED2 Summary of this function goes here
s1 = 1; s2 = 1; s = [s1 s2];
c1 = 0.5; c2 = 0.5; c = [c1, c2];
K = 0.92;
sigma1 = 0.2; sigma2 = 0.2; sigma = [sigma1, sigma2];
corr = 0.3;
rf = 0.02;
maturity = 1;

iter = 10000000;
x = randn(2,iter);
alpha = x - repmat((rf - sigma'.^2 / 2)*(maturity), 1, iter);
theta =1;
phi = max(theta*(sum(repmat(s',1,iter).*exp(x).*repmat(c',1,iter)) - K), 0);
result = phi .* exp(-1/2* dot(alpha,(A \ alpha))) * 2 * pi .* exp(1/2 * (x(1,:)
    .^2 + x(2,:).^2));
int = mean(result);
V = exp(-rf*maturity) / ((2*pi)^(2/2)*abs(det(A))^(1/2))*int;

end
```

Part 1(errorcheck.m)

```
s1 = 1; s2 = 1; s = [s1 s2];
c1 = 0.5; c2 = 0.5; c = [c1, c2];
K = 0.92;
sigma1 = 0.2; sigma2 = 0.2; sigma = [sigma1, sigma2];
corr = 0.3;
rf = 0.02;
maturity = 1;


%  Assuming only 2 stocks s = 2*1, t = 1*1;

M = diag(sigma);
C = zeros(2,2);
C(1,1) = 1; C(1,2) = corr; C(2,1) = corr; C(2,2) = 1;
A = maturity * (M * C * M);

[eV, eD] = eig(A);
v0 = pricing(A);
v = zeros(10,2);
eps = 0.0001;
for i = 1:100
   v(i,2) = abs(pricing(eV * (eD + diag([eps*i, eps*i])) * eV') - v0);
end
v(:, 1) = eps*(1:100);
plot(v(:,1), v(:, 2))

export(v,'File','error.csv','Delimiter',',')
writetable(v,'myData.csv','Delimiter',',')
```

Part 2(errorF.m)

```
function [ error sig ] = error_F( a )
%check the error based on a
K = 1;rf = 0.02;maturity = 1;s =1;


iter = 1000000;
z = randn(1,iter);

m = (rf-a^2/2)*maturity;
x = (z) * maturity^(1/2)*a + m;
F = max(K - s * exp(x), 0) ;
F_m = max(K - s * exp(m), 0);
%P_a = (a^2)^(-1/2) * exp(-(x-m).^2 /(2*a^2*maturity));

%int = (F.*P_a).*exp(x.^2 / 2);
```

```
 sig = F_m;
error = mean(F) - F_m;
end
```

Part 2(ploterror.m)

```
% 2_1
as = 4 : -0.01 :0.0001;
errors = zeros(1,5*80);
errorse = zeros(1,5*80);
for i = 1:(5*80)
    [errors(i) errorse(i)] = error_F(as(i));

end
figure
plot(as, errors, as, errorse);
```